Prime Divisors in an Arithmetic Progression and Explicit ABC Conjecture

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by

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DECLARATION

I, **Vishal Bhardwaj**, Enrollment No. **20308006**, a student in the **Department of Mathematics** of the M.Sc. Program of NIT Manipur, hereby declare that the work presented the project report entitled **" *Prime Divisors in an Arithmetic Progression and Explicit ABC Conjecture*"** contains my ideas in my own words and it has not been submitted for any degree before. I also declare that this project report is a review article by nature and when thoughts and comments are borrowed from other sources, proper references have been cited as applicable.

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CERTIFICATE

It is certified that the work contained in the dissertation titled " ***Prime Divisors in an Arithmetic Progression and Explicit ABC Conjecture***", by **Vishal Bhardwaj**, Enrollment No. **20308006** has been carried out under my supervision for the partial fulfillment of the requirements for awarding the degree of **Master of Science** by **NIT Manipur.** This work has not been submitted elsewhere for a degree.

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Dedicated to my Parents,

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Vishal Bhardwaj

Contents

Introduction

An old and well-known theorem of Sylvester for consecutive integers [1] states that a product of k consecutive integers, each of which exceeds k, is divisible by a prime greater than k.

First, we give some notations that will be used throughout the project.

Let p_i denote the i^{th} prime number. Thus $p_1 = 2$, $p_2 = 3$, $\cdot \ \cdot \ \cdot$. We always write p for a prime number. For an integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime divisors of ν and the greatest prime factor of ν, respectively.

Further, we put $\omega(1) = 0$ and $P(1) = 1$. For positive real number ν and integers l, d with $d \geq 1$, $gcd(l, d) = 1$, we denote

$$\pi(\nu) = \sum_{p \leq \nu} 1,$$

$$\pi_d(\nu) = \sum_{\substack{p \leq \nu \\ gcd(p,d)=1}} 1,$$

$$\pi_d(\nu, d, l) = \sum_{\substack{p \leq \nu \\ p \equiv l (mod\ d)}} 1.$$

We say that a number is effectively computable if it can be explicitly determined in terms of given parameters. We write computable number for an effectively computable number. Let $d \geq 1$, $k \geq 2$, $n \geq 1$ and $y \geq 1$ be integers with $gcd(n, d) = 1$. We denote by

$$\Delta = \Delta(n, d, k) = n(n + d) \cdot \cdot \cdot (n + (k - 1)d)$$

and we write

$$\Delta(n, k) = \Delta(n, 1, k).$$

Further for $x \geq k$, we write

$$\Delta^{'} = \Delta^{'}(x, k) = \Delta(x - k + 1, k).$$

In the above notation, Sylvester's theorem can be stated as

$$P(\Delta(n, k)) > k \ \ if \ \ n > k. \tag{1}$$

On the other hand, there are infinitely many pairs (n, k) with $n \leq k$ such that $P(\Delta) \leq k$. We observe that (1) is equivalent to

$$\omega(\Delta(n, k)) > \pi(k) \;\; if \;\; n > k. \tag{2}$$

Here we notice that

$$\omega(\Delta(n, k)) > \pi(k)$$

since $k!$ divides Δ Let $d > 1$. Sylvester [1] proved that

$$P(\Delta) > k \;\; if \;\; n \geq k + d. \tag{3}$$

Note that (3) includes (1). Langevin [2] improved (3) to

$$P(\Delta) > k \;\; if \;\; n > k$$

Finally Shorey and Tijdeman [3] proved that

$$P(\Delta) > k \;\; unless \;\; (n, d, k) = (2, 7, 3) \tag{4}$$

In Chapter 1, we see the definitions of Elementary Number Theory [4].
In Chapter 2, we see the proof of Sylvester's Theorem. The proof is due to Erdos [5], and the simplifications have been made. This proof is elementary and self-contained; it does not use the prime number theory results. It collects specific estimates on the π function and other functions involving primes.
Chapter 3 gives a brief survey on refinements and generalizations of Sylvester's Theorem. These include the statements of the results of [Laishram and Shorey [6, 7]] We state here two of following results (i) and (ii) appeared.
(i) Let $n > k$. Then $\omega(\Delta(n, k)) \geq \pi(k) + [\frac{3}{4}\pi(k)] - 1$ except when (n, k) belongs to an explicitly given finite set [Laishram and Shorey [6]].
(ii) Let $d > 1$. Then $\omega(\Delta) \geq \pi(2k) - 1$ except when $(n, d, k) = (1, 3, 10)$.[Laishram and Shorey [7]] This is best possible for $d = 2$ since $\omega(1 \cdot 3 \cdot \; \cdot \; \cdot (2k{-}1)) = \pi(2k){-}1$. The latter result (ii) solves a conjecture of Moree [8].
In Chapter 4, we see the Baker's Explicit abc-Conjecture (Laishram and Shorey [9]). The results for the proof of Explicit abc- Conjecture is followed to find the values of described table.

Chapter 1

Historical Remarks

Three positive integers a,b, and c do not satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than two. — *Pierre de Fermat (1637)*

Algebraic number theory has been widely studied since 500 BC when the Pythagorean theorem was first introduced. It was developed in two different ways. One for the Fermat equations, and the other for class field theory. In either way, we have the same purpose: solving Diophantine equations. A Diophantine equation (named after Diophantus of Alexandria) is a polynomial equation in two or more unknowns, such that only the integer solutions are studied. One of the easiest Diophantine equations we have is

$$X^2 + Y^2 = Z^2$$

which is related to the Pythagorean theorem. Infinitely many integral solutions have been found for this equation such as

$$(3, 4, 5), (5, 12, 13), (8, 15, 17), \cdots$$

which we call them Pythagorean triples. One of the most famous and interesting Diophantine equations in the history of mathematics is

$$x^n + y^n = z^n$$

where n is a positive integer. Pierre de Fermat claimed that there are no integral solutions to the Diophantine equation above when $n \geq 3$. This is called the "Fermat's Last Theorem." This theorem was first conjectured in 1637. Fermat claimed that he had a proof, but he did not show it to the public. This problem was left unsolved for more than 350 years until the first successful proof was released in 1994 by Andrew Wiles. Hence Fermat's Last Theorem shows that solving a

Diophantine equation can be extremely difficult. The best possible situation for solving Diophantine equations is when we work over a unique factorization domain. The complexity of calculation is simpler than the equations without unique factorization domains. However, this only means that the calculation is simpler than the other; it still can be extremely difficult.
In this chapter, we review some of the basic definitions and theorems in elementary number theory and abstract algebra courses. In particular, we focus on primes, congruence's, and residues.

1.1 Elementary Number Theory

In this section, we will discuss the basic definition and concepts which will be used throughout the work. In particular, we are discussing Euclid Theorem on Primes and Legendre Symbols [4].

Definition 1.1 *If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that $b = ac$, and we write $a \mid b$. If a does not divide b, we write $a \nmid b$.*

Definition 1.2 (Greatest Common Divisor) *The greatest common divisor of two integers a and b, which are not both 0, is the largest positive integer that divides both a and b. It is written as $gcd(a, b)$.*

Definition 1.3 (Relatively Prime) *The integers a and b, with $a, b \neq 0$, are relatively prime if $gcd(a, b) = 1$.*

Definition 1.4 (Euclid) *There are infinitely many primes.*
Around 300 B.C., Euclid proved that there are an infinite number of prime numbers. The proof is classical and we can explain it to high school students. Suppose that there are only finitely many, $p_1, p_2, \cdots, p_k$ say, then the number $p_1 p_2 \cdots p_k + 1$ is not divisible by any of $p_1, p_2, \cdots, p_k$ and hence must either be prime or divisible by a prime not in our list. This contradiction forces an infinitude of prime numbers, provided there is at least one.

Definition 1.5 *Let m be a positive integer. If a and b are integers, we say that a is congruent to b modulo m, denoted $a \equiv b (mod\ m)$ if $m \mid (a-b)$.*
Example: *We have $5 \equiv 2 \ (mod\ 3)$, since $3 \mid (5-2)$. However, $8 \not\equiv 2 \ (mod\ 5)$ since $5 \nmid (8-2)$.*

Definition 1.6 (Order of a modulo m) *Let m be a positive integer greater than 1, and a an integer relative prime to m. The* order of a modulo m, *denoted by $ord_m(a)$, is the smallest positive integer d such that* $\mathrm{a}^d \equiv 1(mod\ m)$.

Example: $If\ m = 7$ *and* $a = 2$, *then* $2^0 = 1\ (mod\ 7)$,

$$2^1 = 2\ (mod\ 7),$$

$$2^2 = 4\ (mod\ 7),$$

$$2^3 = 1\ (mod\ 7),$$

and 2 has order 3.

1.2 Legendre Symbol

Definition 1.7 *Fix a prime p. Then for any integer a, the Legendre symbol is defined by*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \textit{if a is quadratic residue (mod p)}; \\ -1 & \textit{if a is quadratic nonresidue (mod p);} \\ 0 & \textit{if } p/a. \end{cases}$$

Lemma 1. [4] Let p be a prime and $a \neq 0$. Then $x^2 \equiv a\ (mod\ p)$ has a solution if and only if $a^{(p-1)/2} \equiv 1\ (mod\ p)$.

Proof. For the proof of the forward direction, suppose that $x^2 \equiv a\ (mod\ p)$ has a solution. Let x_0 be this solution, $i.e.$, ${x_0}^{(p-1)/2} \equiv a\ (mod\ p)$.

The last congruence follows from Fermat's Little Theorem.

Conversely, note that $a \not\equiv 0\ (mod\ p)$. So $a\ (mod\ p)$ can be viewed as an element of $(\mathbb{Z}/p\mathbb{Z})^*$, the units of $(\mathbb{Z}/p\mathbb{Z})$. Since $(\mathbb{Z}/p\mathbb{Z})^*$ is a cyclic group, there exists some generator g such that $\langle g \rangle = (\mathbb{Z}/p\mathbb{Z})^*$. So, $a = g^k$, where $1 \leq k \leq p{-}1$. From our hypothesis,

$$a^{(p-1)/2} \equiv g^{k(p-1)/2} \equiv 1\ (mod\ p).$$

Because the order of g is $(p{-}1), (p{-}1)|k\frac{(p-1)}{2}$. But this implies that $2|k^{'}$. So, $k = 2k$. Hence we can write $a(mod\ p)$ as $a \equiv g^k \equiv g^{2k^{'}} \equiv g^{(k^{'})2}\ (mod\ p)$. Hence, a is a square mod p, completing the proof.

Theorem 1.[4] For a prime p and any integer a and b,

$$a^{(p-1)/2} \equiv \binom{a}{b}\ (mod\ p).$$

Proof. If $p|a$, the conclusion is trivial. So, suppose p does not divide a. By Fermat's Little Theorem, $a^{p-1} \equiv 1\ (mod\ p)$. We can factor this statement as

$$a^{p-1} \equiv (a^{(p-1)/2} - 1)(a^{(p-1)/2} + 1) \equiv 0\ (mod\ p).$$

Thus, $a^{(p-1)/2} \equiv \pm 1\ (mod\ p)$. We will consider each case separately.
If $a^{(p-1)/2} \equiv\ (mod\ p)$, then by the previous lemma, there exists a solution to the equation $x^2 \equiv a\ (mod\ p)$. But that would mean that

$$\binom{a}{p} = 1.$$

Otherwise, if $a^{(p-1)/2} \equiv -1\ (mod\ p)$, then by the previous lemma, there is no solution to the equation $x^2 \equiv a\ (mod\ p)$. But that would mean that

$$\binom{a}{p} = -1$$

Hence we conclude that

$$a^{(p-1)/2} \equiv \binom{a}{b}\ (mod\ p).$$

Theorem 2. [4] For a prime p and any integer a and b,

$$\binom{ab}{p} = \binom{a}{p}\binom{b}{p}$$

Proof. We will use Theorem (1) to prove this result, Thus,

$$\binom{ab}{p} = (ab)^{(p-1)/2}\ (mod\ p)$$

Similarly,

$$\binom{a}{p} = (a)^{(p-1)/2}\ (mod\ p)$$

and,

$$\binom{b}{p} = (b)^{(p-1)/2}\ (mod\ p)$$

But then,

$$\binom{ab}{p} = (ab)^{(p-1)/2} = (a)^{(p-1)/2}(b)^{(p-1)/2}\ (mod\ p) = \binom{a}{p}\binom{b}{p}\ (mod\ p)$$

Thus,

$$\binom{ab}{p} = \binom{a}{p}\binom{b}{p}$$

which is what we wanted to show.

Now in the next theorem, we will state some more properties of the Legendre symbol.

Theorem 3. For all odd primes p,

$$\binom{-1}{p} = \begin{cases} 1 & \text{if p} \equiv 1 \text{ (mod 4);} \\ -1 & \text{if p} \equiv 3 \text{ (mod 4).} \end{cases}$$

i.e.

$$\binom{-1}{p} = (-1)^{(p-1)/2}$$

Also,

$$\binom{2}{p} = \begin{cases} 1 & \text{if p} \equiv \pm\ 1 \text{ (mod 8);} \\ -1 & \text{if p} \equiv 3,\ 5 \text{ (mod 8).} \end{cases}$$

Now, we state the famous Law of Quadratic Reciprocity which can be proved using properties of the Legendre symbol and Gauss sums.

Theorem 4. [4] (Law of Quadratic Reciprocity [page 87 [4])] Let p and q be odd primes. Then,

$$\binom{p}{q} = \binom{q}{p}(-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

Definition 2. Let b be a positive odd integer, and suppose that b = $b_1 \cdots b_l$, a product of (not necessarily distinct) primes. For an integer a relatively prime to b, the Jacobi symbol $z\binom{a}{b}$ is defined to be the product

$$\binom{a}{b} = \binom{a}{b_1}\cdots\binom{a}{b_l}$$

where $\binom{a}{b_i}$, $i = 1, 2, \cdots l$ denotes the Legendre symbol. If $b = 1$, $\binom{a}{b} = 1$. We now state some properties of the Jacobi symbol (for a proof, refer to [4])

Theorem 5. (Properties of the Jacobi symbol)

(1) If b is a prime, the Jacobi symbol $\binom{a}{b}$ is the Legendre symbol $\binom{a}{b}$

(2) If $\binom{a}{b} = -1$, then a is not a quadratic residue (mod b). The converse need not hold if b is not a prime.

(3) $\binom{aa'}{bb'} = \binom{a}{b}\binom{a}{b'}\binom{a}{b'}\binom{a'}{b'}$ if aa^i and bb^i are relatively prime.

(4) $\binom{a^2}{b} = \binom{a}{b^2} = 1$ if a and b are relatively prime.

(5) $\binom{-1}{b} = (-1)^{(b-1)/2} = 1$ if $b \equiv 1(mod\ 4)$ and $= -1$ if $\equiv -1(mod\ 4)$

(6) $\binom{2}{b} = (-1)^{(b^2-1)/8} = 1$ if $b \equiv \pm 1(mod\ 8)$ and $= -1$ if $b \equiv \pm 3$ (mod 8)

(7) If a and b are relatively prime odd positive integers, then

$$\binom{a}{b} = \binom{b}{a}(-1)^{\frac{a-1}{2}\frac{b-1}{2}}$$

(For a proof, see pg. 82, 83 of [4]).

Chapter 2

Sylvester's Theorem for Consecutive Integers

In this chapter, we consider the Theorem of Sylvester [1] for consecutive integers stated in the Introduction, see (1).

Theorem 1.0.1. Let $d = 1$. Then

$$P(\Delta) > k \;\; if \;\; n > k. \tag{1.0.1}$$

In Laishram and Shorey [7] consider $n \leq k$. For $1 \leq n \leq p_{\pi(k)+1} - k$ where $p_{\pi(k)+1}$ is the smallest prime exceeding k, we see that $P(\Delta) \leq k$ since $n + k - 1 < p_{\pi(k)+1}$. Thus it is necessary to assume $n > p_{\pi(k)+1-k}$ for the proof of $P(\Delta) > k$. Then $n = p_{(\pi(k)+1} - k + r$ for some $1 \leq r < k$ and hence $p_{\pi(k)+1} = n + k - r$ is a term in Δ, giving $P(\Delta) > k$.

For $x \geq 2k$, $x = n + k - 1$ and a prime $p > k$, we see that p divides $\binom{x}{k}$ if and only if p divides $\Delta = \Delta(n, k)$. Thus we observe that (1.0.1) is equivalent to the following result.

Theorem 1.0.2. If $x \geq 2k$, then $\binom{x}{k}$ contains a prime divisor greater than k.

Therefore, we shall consider Theorem 1.0.2. The proof is due to Erdös [1]. This proof is elementary and self-contained; it does not use the prime number theory results.

2.1 Lemmas for the Proof of Theorem 1.0.2

Lemma 1.1.1. [7] Let X be a positive real number and k_0 a positive integer. Suppose that $p_{i+1} - p_i < k_0$ for any two consecutive primes $p_i < p_{i+1} \leq p_{\pi(X)+1}$. Then $P(x(x-1) \cdots (x-k+1)) > k$ for $2k \leq x < X$ and $k \geq k_0$.

Proof. Let $2k \leq x < X$. We may assume that none of x, $x-1$, $\cdots, x-k+1$ is a prime, since otherwise the result follows. Thus

$$p_{\pi(x-k+1)} < x-k+1 < x < p_{\pi(x-k+1)+1}\ p_{\pi(X)+1}.$$

Hence by our assumption, we have

$$k-1 = x-(x-k+1) < p_{\pi(x-k+1)+1} - p_{\pi(x-k+1)} < k_0,$$

which implies $k-1 < k_0-1$, a contradiction.

Lemma 1.1.2. [[7]] Suppose that Theorem 1.0.2 holds for all primes k, then it holds for all k.

Proof. Assume that Theorem 1.0.2 holds for all primes k. Let $k_1 \leq k < k_2$ with k_1, k_2 consecutive primes. Let $x \geq 2k$. Then $x \geq 2k_1$ and $x(x-1)\cdots(x-k_1+1)$ has a prime factor $p > k_1$ by our assumption. Further, observe that $p \geq k_2 > k$ since k_1 and k_2 are consecutive primes. Hence p divides $\frac{x\cdots(x-k_1+1)(x-k_1)\cdots(x-k+1)}{k!} = \binom{x}{k}$.

By Lemma 1.1.2, it is enough to prove the Theorem 1.0.2 for k prime, which we assume from now on. Further, take $x \geq 2k$.

Lemma 1.1.3. [10] Let $p^a \mid \binom{x}{k}$. Then $p^a \leq x$.

Proof. It is observed that

$$ord_p\binom{x}{k} = \sum_{\nu=1}^{\infty}\left(\left[\frac{x}{p^\nu}\right] - \left[\frac{x-k}{p^\nu}\right] - \left[\frac{k}{p^\nu}\right]\right)$$

Each of the summand is at most 1 if $p \leq x$ and 0 otherwise. Therefore $ord_p\binom{x}{k} \leq s$ where $p^s \leq x < p^{s+1}$. Thus

$$p^a \leq p^{ord_p\binom{x}{k}} \leq p^s \leq x \qquad (1.1.1)$$

Lemma 1.1.4. [10] For $k > 1$, we have

$$\binom{2k}{k} > \frac{4^k}{2\sqrt{k}} \qquad (1.1.2)$$

and

$$\binom{2k}{k} < \frac{4^k}{\sqrt{2k}} \qquad (1.1.3)$$

Proof. For k > 1, we have

$$1 > \left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\ldots\left(1-\frac{1}{(2k-1)^2}\right) = \frac{2\cdot 4}{3^2}\frac{4\cdot 6}{5^2}\ldots\frac{(2k-2)\cdot 2k}{(2k-1)^2}$$

$$> \frac{1}{4k}\left(\frac{2^k k!}{3\cdot 5\cdots(2k-1)}\right)^2 = \frac{1}{4k}\left(\frac{4^k(k!)^2}{(2k)!}\right)^2$$

implying (1.1.2). Further we have

$$1 > \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)...\left(1-\frac{1}{(2k-2)^2}\right) = \frac{1\cdot 3}{2^2}\frac{3\cdot 5}{4^2}...\frac{(2k-3)\cdot(2k-1)}{(2k-2)^2}$$

$$> \frac{1}{2k-1}\left(\frac{3\cdot 5\cdots(2k-1)}{2^k k!}\right)^2 > \frac{4k^2}{2k}\left(\frac{2k!}{4^k(k!)^2}\right)^2$$

implying (1.1.3)

Lemma 1.1.5. [10] We have

$$\prod_{p\le x} p \prod_{p\le\sqrt{x}} p \prod_{p\le\sqrt[3]{x}} p \cdot\cdot\cdot < 4^x$$

Proof. We see that for every prime p and a positive integer a with

$$x < p^a \le 2x,$$

we have,

$$ord_p\binom{2x}{x} = ord_p\left(\frac{(2x!)}{(x!)^2)}\right) = \sum_{\nu=1}^{\infty}\left(\left[\frac{2x}{p^i}\right] - 2\left[\frac{2x}{p^i}\right]\right) \ge 1 \qquad (1.1.5)$$

since

$$\left[\frac{2x}{p^i}\right] - 2\left[\frac{x}{p^i}\right] \ge 0 \;\; and \;\; \left[\frac{2x}{p^a}\right] - \left[\frac{x}{p^a}\right] = 1.$$

Let $\lceil\nu\rceil$ denote the least integer greater than or equal to ν. Let $2^{m-1} \le x < 2^m$ and we put

$$a_1 = \lceil\frac{x}{2},\rceil, a_2 = \lceil\frac{x}{2^2}\rceil, \cdots, a_h = \lceil\frac{x}{2^h}\rceil, \cdots, a_m = \lceil\frac{x}{2^m}\rceil = 1.$$

Then

$$a_1 > a_2 > \cdots > a_m$$

and

$$a_h < \frac{x}{2^h} + 1 = \frac{2x}{2^{h+1}} + 1$$

implies

$$a_h \le 2a_{h+1} + 1$$

Also, we have $2a_2 < \frac{x}{2} + 2 \le a_1 + 1$. Therefore

$$2a_2 \le a_1 + 1 \qquad (1.1.6)$$

Since $2a_1 \geq x$, we see that

$$(1, x] \cup_{h=1}^{m} (a_h, 2a_h].$$

Let p and r be given such that $p^r \leq x < p^{r+1}$. Let $1 \leq i \leq r$. Then $p^i \leq x$. It is clear from the above inclusion that there exists k_i such that

$$a_{k_i} < p^i \leq 2a_{k_i}$$

Observe that $a_{k_i} \neq a_{k_j}$ for $1 \leq j < i \leq r$ since $pa_{k_j} < p_{j+1} \leq p^i \leq 2a_{k_i}$. Thus we see from (1.1.5) that

$$p^r \mid \binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m}$$

Hence we have

$$\prod_{p\leq x} p \prod_{p\leq\sqrt{x}} p \prod_{p\leq\sqrt[3]{x}} p \cdots = \prod_{p^r\leq x<p^{r+1}} \leq \binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m}$$

the middle product being taken over all prime powers p^r with $p^r\ x < p^{r+1}$. To complete the proof of the lemma, we show that

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < 4^x. \tag{1.1.7}$$

By direct calculation, we check that (1.1.7) holds for $x \leq 10$. For example, when $x = 5$, we have $a_1 = 3, a_2 = 2, a_3 = 1$ so that

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_3}{a_3} = 20 * 6 * 2 < 4^5.$$

Suppose that $x > 10$ and (1.1.7) holds for any integer less than x. Then

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < \binom{2a_1}{a_1}4^{2a_2-1} \tag{1.1.8}$$

which we obtain by applying (1.1.7) with $x = 2_{a_2}{-}1$ and seeing that

$$\lceil\frac{1}{2}(2a_2-1),\rceil = a_2, \lceil\frac{1}{4}(2a_2-1),\rceil = \lceil\frac{a_2}{2}\rceil = a_3. \cdots.$$

We obtain from (1.1.3) that

$$\binom{2x}{x} < 4^{x-1}$$

for $x \geq 8$. Hence we see that

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < 4^{a_1-1+2a_2-1} \qquad (1.1.9)$$

Lemma 1.1.6. [10] Assume that

$$P\left(\binom{x}{k}\right) \leq k \qquad (1.1.10)$$

holds. Then we have

(i) $x < k^2$ *for* $k \geq 11$.

(ii) $x < k^{\frac{3}{2}}$ *for* $k \geq 37$

Proof: We have

$$\binom{x}{k} = \frac{x}{k}\frac{x-1}{k-1}\cdots\frac{x-k+1}{1} > \binom{x}{k}^k$$

From $P\left(\binom{x}{k}\right) \leq k$ and $p^a \mid \binom{x}{k}$, we get $\binom{x}{k} \leq x^{\pi(k)}$. Comparing the bounds for $\binom{x}{k}$, we have

$$\frac{x^k}{k^k} < \binom{x}{k} \leq x^{\pi(k)} \;\; implies \;\; x < k^{\frac{k}{k-\pi(k)}}$$

From (1.1.10) and Lemma 1.1.3, we have $\binom{x}{k} \leq x^k$. Comparing the upper and lower bound for $\binom{x}{k}$, we derive that

$$x < k^{\frac{k}{k-\pi(k)}}. \qquad (1.1.11)$$

For $k \geq 11$, we exclude 1 and 9 to see that there are at most $[\frac{k+1}{2}]-2$ odd primes upto k. Hence $\pi(k) \leq 1 + [\frac{k+1}{2}]- - 2 \leq \frac{k}{2}$ for $k \geq 11$ giving $x < k^2$ for $k \geq 11$. Further the number of composite integers $\leq k$ and divisible by 2 or 3 or 5 is

$$a_1 = \lceil\frac{k}{2}\rceil + \lceil\frac{k}{3}\rceil + \lceil\frac{k}{5}\rceil - \lceil\frac{k}{6}\rceil - \lceil\frac{k}{10}\rceil - \lceil\frac{k}{15}\rceil + \lceil\frac{k}{30}\rceil - 3 \geq$$
$$\lceil\frac{k}{2}\rceil + \lceil\frac{k}{3}\rceil + \lceil\frac{k}{5}\rceil + \lceil\frac{k}{30}\rceil - \lceil\frac{k}{6}\rceil - \lceil\frac{k}{10}\rceil - \lceil\frac{k}{15}\rceil - 7$$
$$= \frac{11}{15}k - 7$$

Thus we have $\pi(k) \leq k - 1 - (\frac{11}{15}k - 7)$ for $k \geq 90$. . By direct computation, we see that $\pi(k) \leq \frac{k}{3}$ for $37 \leq k < 90$. Hence

$$\frac{k}{k-\pi(k)} = \begin{cases} 2 & \text{for } k \geq 11 \\ \frac{3}{2} & \text{for } k \geq 37 \end{cases}$$

which, together with (1.1.11), proves the assertion of the lemma.

Chapter 3

A Survey of Refinements and Extensions of Sylvester's Theorem

Let n, d and $k \geq 2$ be positive integers. For a pair (n, k) and a positive integer h, we write $[n, k, h]$ for the set of all pairs $(n, k), \cdot \cdot \cdot, (n + h{-}1, k)$ and we set $[n, k] = [n, k, 1] = (n, k)$. Let $W(\Delta)$ denote the number of terms in Δ divisible by a prime $> k$. We observe that every prime exceeding k divides at most one term of Δ. On the other hand, a term may be divisible by more than one prime exceeding k. Therefore we have

$$W(\Delta) \leq \omega(\Delta){-}\pi_{d(k)} \qquad (3.0.1)$$

If $max(n, d) \leq k$, we see that $n + (k{-}1)d \leq k^2$ and therefore no term of Δ is divisible by more than one prime exceeding k. Thus

$$W(\Delta) = \omega(\Delta) - \pi_d(k) \;\; if \;\; max(n, d) \leq k \qquad (3.0.2)$$

We are interested in finding lower bounds for $P(\Delta)$, $\omega(\Delta)$ and $W(\Delta)$. The first result in this direction is due to Sylvester [1] who proved that

$$P(\Delta) > k \;\; if \;\; n \geq d + k. \qquad (3.0.3)$$

This immediately gives

$$\omega(\Delta) \geq \pi_d(k) \;\; if \;\; n \geq d + k. \qquad (3.0.4)$$

We give a survey of several results in this direction.

3.1 Improvements of $\omega(\Delta(n,k)) > \pi(k)$

Let $d = 1$. A proof of Sylvester's result is given in Chapter 1. The result of Sylvester was rediscovered by Schur and *Erdos* [5]. Let $k = 2$ and $n > 2$. We see that $\omega(n(n+1)) \neq 1$ since $gcd(n, n+1) = 1$. Thus $\omega(n(n+1)) \geq 2$.

Suppose $\omega(n(n+1) = 2$. Then both n and $n+1$ are prime powers. If either n or $n+1$ is a prime, then $n+1$ or n is a power of 2, respectively. A prime of the form $2^{2^m}+1$ is called a Fermat prime and a prime of the form $2^m - 1$ is called a Mersenne prime. Thus we see that either n is a Mersenne prime or $n+1$ is a Fermat prime. Now assume that $n = p^\alpha, n+1 = q^\beta$ where p, q are distinct primes and $\alpha, \beta > 1$. Thus $q^\beta - p^\alpha = 1$, which is Catalan equation. Thus n = 8 is the only other n for which $\omega(n(n+1)) = 2$. For all other n, we have $\omega(n(n+1)) \geq 3$. Let $k \geq 3$. We observe that in Laishram and Shorey [6].

$$\omega(\Delta(n,k)) = \pi(2k) \ if \ n = k+1 \qquad (3.1.1)$$

If $k+1$ is prime and $2k+1$ is composite, then from (3.1.1) by writing

$$\Delta(k+2,k) = \Delta(k+1,k)\frac{2k+1}{k+1}$$

that

$$\omega(\Delta(k+2,k)) = \pi(2k) - 1. \qquad (3.1.2)$$

Let $k+1$ be a prime of the form $3r+2$. Then $2k+1 = 3(2r+1)$ is composite. Since there are infinitely many primes of the form $3r+2$, Laishram and Shorey [6] that there are infinitely many k for which $k+1$ is prime and $2k+1$ is composite. Therefore (3.1.2) is valid for infinitely many k. Thus an inequality sharper than $\omega(\Delta(n,k)) \geq \pi(2k) - 1$ for $n > k$ is not valid.

Saradha and Shorey [11] [Corollary 3] extended the proof of Erdos [5] given in Chapter 2 to sharpen (3.0.4) and gave explicit bounds of $\omega(\Delta(n,k))$ as

$$\omega(\Delta(n,k)) \geq \pi(k) + \left[\frac{1}{3}\pi(k)\right] + 2 \ \ if \ \ n > k > 2$$

unless $(n,k) \in S_1$ where S_1 is the union of sets ([4,3], [6, 3, 3], [16, 3], [6, 4], [6, 5, 4], [12, 5], [14, 5, 3], [23, 5, 2], [7, 6, 2], [15, 6], [8, 7, 3], [12, 7], [14, 7, 2], [24, 7], [9, 8], [14, 8], [14, 13, 3], [18, 13], [20, 13, 2], [24, 13], [15, 14], [20, 14], [20, 17]). (3.1.4)

(Laishram and Shorey [6]) improved it to $\frac{3}{4}$. Define

$$\delta(k) = \begin{cases} 2 & \text{if } 3 \le k \le 6 \\ 1 & \text{if } 7 \le k \le 16 \\ 0 & otherwise \end{cases}$$

so that

$$\left[\frac{3}{4}\pi(k)\right] + \delta(k) \ge \left[\frac{1}{3}\pi(k)\right] + 2.$$

We have Theorem 3.1.1. Let $n > k \ge 3$. Then

$$\omega(\Delta(n,k)) \ge \pi(k) + \left[\frac{3}{4}\pi(k)\right] - 1 + \delta(k) \tag{3.1.5}$$

unless

$$(n,k) \in S_1 \cup S_2$$

where S_1 is given by (3.1.4) and S_2 is the union of sets. We note here that the right hand sides of (3.1.3) and (3.1.5) are identical for $3 \le k \le 18$. Theorem 3.1.1 is an improvement of (3.1.3) for $k \ge 19$. The proof of this Theorem uses sharp bounds of the π function due to Dusart given by Lemma 2.0.2. We derive the following two results from Theorem 3.1.1. We check that the exceptions in Theorem 3.1.1 satisfy $\omega(\Delta(n,k)) \ge \pi(2k)-1$.

Corollary 3.1.2. Let $\epsilon > 0$ and $n > k$. Then there exists a computable number k_0 depending only on ϵ such that for $k \ge k_0$, we have

$$\omega(\Delta(n,k)) \ge (2-\epsilon)\pi(k).$$

We end this section with a conjecture of Grimm [11]:

Suppose $n, n+1, \cdots, n+k-1$ are all composite numbers, then there are distinct primes p_{i_j} such that $p_{i_j} \mid (n+j)$ for $0 \le j < k$.

This conjecture is open. The conjecture implies that if $n, n+1, \cdots, n+k-1$ are all composite, then $\omega(\Delta(n,k)) \ge k$ which is also open. Let $g(n)$ be the largest integer such that there exist distinct prime numbers $P_0, \cdots P_{g(n)}$ with $P_i \mid n+i$. A result of Ramachandra, Shorey and Tijdeman [12] states that $g(n) > c_1 \left(\frac{\log n}{\log\log n}\right)$ where $c_1 > 0$ is a computable absolute constant. Further Ramachandra, Shorey and Tijdeman [13] showed that

$$\omega(\Delta(n+1,k)) \ge k \;\; for \;\; 1 \le k \le \exp\left(c_2(\log n)^{\frac{1}{2}}\right)$$

where c_2 is a computable absolute constant.

Chapter 4

Baker's Explicit abc-Conjecture

The abc-Conjecture is one of the most interesting recent Conjectures in number theory. The past decade or so has been marked by great progress in number theory, and concurrent statements of Conjectures the resolution of which would seem to require further steps forward. The abc-Conjecture sits among this group of statements in a web of equivalences and implications which give, if nothing else, at least heuristic evidence for their truth. The goal of this dissertation is to highlight some of the many implications of this Conjecture; in the process much interesting mathematics is discussed. We begin by stating the Conjecture. Like many other famous Conjectures in number theory, it appears innocuous.
The Conjecture of Masser-Oesterle, popularly known as abc-Conjecture, has many consequences. We use an explicit version due to Baker to solve Explicit abc - Conjecture.

4.1 Introduction

In this section we will see Mason's Inequality, the polynomial version of the ABC Conjecture
Mason Theorem: [**page 2** [14]] Let $A(x), B(x), C(x)$ be polynomials in $\mathbb{C}[x]$ are nonzero, relatively prime polynomials, not all constant, and if

$$A + B = C$$

then

$$max\{degA, degB, degC\} \leq |N(ABC)| - 1.$$

Mason's Inequality can be used to tackle Fermat's Last Equation where the exponents are different.

Generalized Fermat's Equation: [page 12 [14]] There are no co-prime non-constant polynomial solutions to the generalized Fermat equation:

$$A(t)^p + B(t)^q = C(t)^r$$

when

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1. \qquad (i)$$

Proof. By Mason's Inequality

$$max\{p\ degA, q\ degB, r\ degC\} \leq |N(A^pB^qC^r)|{-1} = |N(ABC)|{-1}.$$

By equation (i) we have

$$|N(ABC)| \leq degA + degB + degC$$

$$= \frac{p\ degA}{p} + \frac{qdegB}{q} + \frac{r\ degC}{r}$$

$$\leq \Big(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\Big)max\{p\ degA, q\ degB, r\ degC\}$$

$$\leq max\{p\ degA, q\ degB, r\ degC\}$$

Then we obtain

$$max\{p\ degA, q\ degB, r\ degC \leq max\{p\ degA, q\ degB, r\ degC - 1,$$

which is a contradiction.

In Mason's Inequality, we had no ϵ in the exponent. A natural first thought is to get rid of the ϵ. Will getting rid of the ϵ work?

Suppose that A, B, and C, be integers are each nonzero, relatively prime factor, and if

$$A + B = C$$

For the case of polynomials we had

$$max\{degA, degB, degC\} < |N(ABC)|.$$

So we might conjecture the analogue

$$max\{|A|, |B|, |C|\} < |Rad(ABC).| \qquad (ii)$$

However, (ii) would be false, let's consider

$$1 + 2^3 = 3^2$$

then (ii) would have $9 < 6$;

$$3^3 + 5 = 2^5,$$

would have $32 < 30$;

$$2^5 + 7^2 = 3^4,$$

would have $81 < 42$ and

$$1 + 2^9 = 3^3 * 19$$

would have $513 < 114$.

Clearly, all the above four inequality are not true. Thus (ii) cannot hold.

4.2 ABC Conjecture

Suppose that A, B, and C, be integers are each nonzero, relatively prime factor, and if

$$A + B = C$$

Then for every $\epsilon > 0$ there exist a constant $K(\epsilon) > 0$ such that,

$$max\{|A|, |B|, |C|\} \leq K(\epsilon)\ Rad(ABC)^{1+\epsilon}$$

we will define ABC-triple as a triple (A, B, C) with A, B, C being positive co-prime integers that satisfy $A + B = C$ with $A < B$. $(1, 2, 3)$ would be the smallest example of an ABC-triple [Chapter 2 [14]] .

Second, we will define ABC-hit as an ABC-triple that satisfy $Rad(ABC) < C$. Looking at $(1, 8, 9)$, we can see that is an ABC-hit since $1 + 8 = 9$, gcd(1, 8, 9) = 1 and

$$Rad(1 \cdot 8 \cdot 9) = Rad(1 \cdot 2^3 \cdot 362) = 2 \cdot 3 = 6 < 9$$

A few examples,

$$2 + 3^{10} \cdot 109 = 23^5, Rad(2 \cdot 3^{10} \cdot 109 \cdot 23^5) = 2 \cdot 3 \cdot 23 \cdot 109 = 15042,$$

$$11^2 + 3^2 \cdot 566 \cdot 7^3 = 2621 \cdot 23, Rad(2^{21} \cdot 3^2 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 23) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 53,130$$

Now, For any positive integer $i > 1$, let $N = N(i) = \prod_{p|i} p$ be the radical of i, $P(i)$ be the greatest prime factor of i and $\omega(i)$ be the number of distinct prime factors of i and we put $N(1) = 1$, $P(1) = 1$ and $\omega(1) = 0$. The well known Conjecture of Masser-Oesterl´e states (Laishram and Shorey [9]) that

Conjecture 1.1. Oesterl´e and Masser's abc-Conjecture: For any given $\epsilon > 0$ there exists a computable constant $\mathcal{C}_\epsilon$ depending only on ϵ such that if

$$a + b = c \qquad (1)$$

where a, b, and c are co-prime positive integers, then

$$c \leq \mathcal{C}_\epsilon (\prod_{p|abc} p)^{1+\epsilon}$$

It is known as abc-Conjecture; the name derives from using the letters a, b, and c in (1). The abc−Conjecture has already become well known for the number of interesting consequences and remains as one of the main open problems in number theory. Many famous Conjectures and theorems in number theory would follow immediately from the abc−Conjecture. An explicit version of this Conjecture due to Baker [9] is the following:

Conjecture 1.2. Explicit abc-Conjecture: Let a, b, and c be pairwise coprime positive integer satisfying (1). Then

$$c < \frac{6}{5} N \frac{(\log N)^\omega}{\omega!}$$

where $N = N(abc)$ and $\omega = \omega(N)$. We see when $\omega \in \{0, 1\}$ or N is odd then (1) does not hold. Therefore we always have N even and $\omega \geq 2$ unless $(a, b, c) = (1, 1, 2)$. We observe that $N = N(abc) \geq 2$ whenever a, b, c satisfy (1). We shall refer to Conjecture 1.1 as abc−Conjecture and Conjecture 1.2 as explicit abc−Conjecture.

Conjecture 1.2 implies the next Explicit version of Conjecture 1.1

Theorem 1. (Laishram and Shorey [9]) Assume Conjecture 1.2. Let a, b and c be pairwise coprime positive integers satisfying (1) and $N = N(abc)$. Then we have

$$c < N^{1+\frac{3}{4}}$$

Further for $0 < \epsilon \leq \frac{3}{4}$, there exists ω_ϵ depending only ϵ such that when $N = N(abc) \geq N_\epsilon = \prod_{p \leq p_\epsilon} p$, we have

$$c < \kappa_\epsilon N^{1+\epsilon} \qquad (2)$$

where

$$\kappa_\epsilon = \frac{6}{5\sqrt{2\pi \, max(\omega, \omega_\epsilon)}} \leq \frac{6}{\sqrt{2\pi\omega_\epsilon}}$$

with $w = w(n)$. Here are some values of ϵ, ω_ϵ and N_ϵ

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	75	548	6460
N_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{679.585}$	$e^{1004.763}$	$e^{3894.57}$	e^{63727}

Thus $c < N^2$ which was Conjectured in Granville and Tucker [15].

I thank Professor Shanta Laishram for allowing me to include his proof in the project.

4.3 Results for the Proof of Theorem 1

For an Integer $i > 0$, p_i is the i^{th} prime. For a real $x > 0$, Let $\Theta(x) = \prod_{p \leq x} p$ and $\theta(x) = \log(\Theta(x))$. We write $\log_2 i$ for $\log(\log i)$. We have

Lemma 2.1. We have

(i) $p_i \geq i(\log i + \log_2 i - 1)$ for $i \geq 1$

(ii) $\theta(p_i) \geq i(\log i + \log_2 i - 1.076869)$ for $i \geq 1$

(iii) $\theta(x) < 1.000081x$ for $x > 0$

The estimates (i) is due to Dusart, see [16]. The estimate (ii) is ([17], Theorem 6). For estimate (iii), see [18]. Proof of Theorem 1 by (Laishram and Shorey [9])

4.4 Proof of Theorem 1

Let $\epsilon > 0$ and $N \geq 1$ be an integer with $W(N) = \omega$. Then $N \geq \Theta(p_\omega)$ or $logN \geq \theta(p)$. Given ω, we observe that $\frac{M^\epsilon}{(\log M)^\omega}$ is an increasing function for $logM \geq \frac{\omega}{\epsilon}$. Let

$$X_0(i) = i(\log i + \log_2 i - 1.076869).$$

Then $\theta(p_\omega) \geq \omega X_0(\omega)$ by Lemma 2.1 (iii). Observe that $X_0(i) > 1$ for $i \geq 5$. Let $w_i \geq 4$ be smallest ω such that

$$\epsilon X_0(\omega) - \log X_0(\omega) \geq 1 \ for \ all \ \omega \geq \omega_1 \tag{3}$$

Note that $\epsilon X_0(\omega) \geq 1$ for $\omega \geq \omega_1$ implying $\log N \geq \theta(p_\omega) \geq \omega X_0(\omega) \geq \frac{\omega}{\epsilon}$ for $\omega \geq \omega_1$ by Lemma 2.1 (iii). Therefore

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega! \Theta(p_\omega)^\epsilon}{(\theta(_\omega))^\omega} \geq \frac{\omega! e^{\epsilon\omega X_0(\omega)}}{(\omega X_0(\omega))^\omega} > \sqrt{2\pi\omega}(\frac{\omega}{e})^\omega \frac{e^{\epsilon\omega X_0(\omega)}}{(\omega X_0(\omega))^\omega} \ for \ \omega \geq \omega_1.$$

Thus for $\omega \geq \omega_1$, we have from (3) that

$$\log\left(\frac{\omega! e^{\epsilon\omega X_0(\omega)}}{(\omega X_0(\omega))^\omega}\right) > \log\sqrt{2\pi\omega} + \omega(\log(\omega) - 2) + \epsilon\omega X_o(\omega) - \omega(\log\omega + \log X_o(\omega))$$
$$> \log\sqrt{2\pi\omega} + \omega(\epsilon X_o(\omega) - \log X_o(\omega) - 1) \geq \log\sqrt{2\pi\omega}$$

implying

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega! \Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} \geq \sqrt{2\pi\omega} \ for \ \omega \geq \omega_1.$$

Define ω_ϵ be the smallest $\omega \leq \omega_1$ such that

$$\theta(p_\omega) \geq \frac{\omega}{\epsilon} \ and \ \frac{\omega! \Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} \geq \sqrt{2\pi\omega} \ for \ all \ \omega_\epsilon \leq \omega \leq \omega_1. \tag{4}$$

by taking the exact values of ω and θ. Then clearly

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega! \Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} \geq \sqrt{2\pi\omega} \ for \ \omega \geq \omega_\epsilon. \tag{5}$$

Here are values of ω_ϵ for some ϵ values.

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	75	548	6460

Let $\omega < \omega_\epsilon$ and $N \geq \theta(\omega_\epsilon.)$ Then $\log N \geq \theta(\omega_\epsilon) \geq \frac{\omega_\epsilon}{\epsilon}$. Therefore

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega!\Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} = \frac{\omega_\epsilon!\Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^{\omega_\epsilon}}.\frac{\omega!}{\omega_\epsilon}(\theta(p_{\omega_\epsilon}))^{\omega_\epsilon-\omega} > \sqrt{2\pi\omega_\epsilon}\frac{\omega!{\omega_\epsilon}^{\omega_\epsilon-\omega}}{\omega_\epsilon!} \geq \sqrt{2\pi\omega_\epsilon}.$$

Combining this with (10,) we obtain

$$\frac{(\log N)^\omega}{\omega!} < \frac{N^\epsilon}{\sqrt{2\pi\ max(\omega, \omega_\epsilon)}} \leq \frac{N^\epsilon}{\sqrt{2\pi\omega_\epsilon}}\ for\ N \geq \Theta(\omega_\epsilon) \tag{6}$$

Further

$$\frac{(\log N)^\omega}{\omega!} < \frac{5N^\epsilon}{6}\ for\ N \geq 1. \tag{7}$$

For that let $\epsilon = \frac{3}{4}$. Then $\omega_\epsilon = 14$ and we may assume that $N < \Theta(p_{14})$. Then $\omega = \omega(N) < 14$. Observe that $N \geq \Theta(p_\omega)$ and $\frac{N^{\frac{3}{4}}}{(\log N)^w}$ is increasing for $\log N \geq \frac{4\omega}{3}$.

For $4 \leq \omega < 14$, we check that

$$\theta(p_\omega) \geq \frac{4\omega}{3}\ and\ \frac{\omega!\Theta(p_\omega)^{\frac{3}{4}}}{(\theta(p_\omega))^\omega} > \frac{6}{5}$$

implying (7) when $4 \leq \omega = \omega(N) < 14$. Thus we may assume that $\omega = \omega(N) < 4$.

We check that

$$\frac{\omega! N^{\frac{3}{4}}}{(\log N)^\omega} > \frac{6}{5}\ at\ N = e^{\frac{4w}{3}} \tag{8}$$

for $1 \leq \omega < 4$ implying (7) for $N \geq e^{\frac{4w}{3}}$. Thus we may assume that $N < e^{\frac{4w}{3}}$. Then $N \in \{2, 3\}$ if $\omega = \omega(N) = 1$, $N \in \{6, 10, 12, 14\}$ if $\omega = \omega(N) = 2$ and $N \in \{30, 42\}$ if $\omega(N) = 3$. For these values of N too, we find that (8) is valid implying (7). Clearly (7) is valid when N = 1.

Now, Assume Conjecture 1.2. Let $\epsilon > 0$ be given. Let a, b, c be the positive integers such that $a + b = c$ and $gcd(a, b) = 1$.

By Conjecture 1.2, $c < \frac{6}{5}N\frac{(\log N)^\omega}{\omega!}$ where $N = N(abc)$. Now assertion 2 follows from (7). Let $0 < \epsilon \leq \frac{3}{4}$ and $N_\epsilon = \Theta(p_{\omega_\epsilon})$. By (6), we have

$$c < \frac{6N^{1+\epsilon}}{5\sqrt{2\pi\ max(\omega, \omega_\epsilon)}}$$

The table is obtained by taking the table values of $\epsilon, \omega_\epsilon$ given after (5) and computing N_ϵ for those ϵ given in the table. Hence the Theorem.

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