MAJOR PROJECT REPORT

ON

A STUDY OF HYPERGEOMETRIC FUNCTIONS

AT

GD GOENKA UNIVERSITY, GURUGRAM

submitted in partial fulfilment of the requirements for the award of the degree of

BACHELOR OF SCIENCE HONOURS

IN

MATHEMATICS

Submitted by: Vishal Bhardwaj

Programme: B.Sc.(Hons.) Mathematics

University Roll No.: 160020211003

Supervisor: Dr. Smita Sood

Designation: Assistant Professor

Department: Mathematics

School of Basic & Applied Sciences,

G D Goenka University

MAY, 2019

DECLARATION

I declare that the topic "**A Study of Hypergeometric Functions**" for my major project under **Dr. Smita Sood** has not been submitted by any other institute or university. The complete project report is of my knowledge and done by me.

Place:

Date:

Vishal Bhardwaj

B.Sc. (Hons.) Mathematics

G.D Goenka University

ACKNOWLEDGEMENT

I would like to express my deep appreciation to my guide Dr. Smita Sood, Professor, Department of Mathematics, GD Goenka University, Gurugram for her valuable comments and suggestions time, guidance given me during the project period.

I would like to acknowledge who helped me during my project and helped me become more confident in my newly acquired skills. Special thanks to Manvi Singh for teaching me to be sincere to my goals and also for giving me advice. Thanks to Himanshu Bhardwaj, Chetan Vashistha, Varsha Sharma, Manvi Singh, and Manjeet Sharma for teaching me how to keep calm under pressure and remember to enjoy my learning. Thanks to all my friends and classmates who encourage me a lot in this Project.

Any omission in this brief acknowledgement does not mean lack of gratitude.

Vishal Bhardwaj

Contents

1. ***Introduction***

The study of hypergeometric functions is more than 200 years old. They appear in the work of Euler, Gauss, Riemann, and Kummer. Their integral representations were studied by Barnes and Mellin, and special properties of them by mathematicians. The famous Gauss hypergeometric equation is everywhere in mathematical physics. For one-variable hypergeometric functions this interplay has been well understood for several decades. On the other hand, it is possible to extend each one of these approaches but one may get slightly different results. There has been a great revival of interest in the study of hypergeometric functions. It would be impossible to give even an introduction to this theory. This emphasizes the fact that I have chosen to highlight a number of topics which I hope will make the interested in further study of this beautiful subject, but I have made no attempt to give a comprehensive view of the field. There is no claim of originality in these notes.

A special function is a function having a particular use in mathematical physics or some other branch of mathematics. Prominent examples include the Bessel functions, Gamma functions, Beta functions, Hypergeometric functions. Certain mathematical functions occur often enough in fields like physics and engineering to justify special consideration. They form a class of well-studied functions with an extensive literature and appropriately enough, are collectively called special functions. These functions carry such names as Bessel functions, Gamma functions, Beta functions, Hypergeometric functions and the like. Most of the special functions encountered in such applications have a common root in their relation to the hypergeometric function. The purpose of this research is to establish this relationship and use it to obtain many of the interesting and important properties of the special functions met in applied mathematics.

2. ***Hypergeometric Function***

Definition: Hypergeometric function is denoted by ${}_2F_1(\alpha, \beta; \gamma; x)$ and is defined by

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \frac{x^r}{r!} $$

Where, $2 -$ refers to number of parameters in numerator

$1 -$ refers to number of parameters in denominator

Remark. The series of R.H.S is $1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \cdots$

is called the Gauss Series or the ordinary hypergeometric series. It is represented by the symbol

${}_2F_1(\alpha, \beta; \gamma; x)$

In particular, if $\alpha = 1, \beta = \gamma$, then the above series become,

$1 + x + x^2 + x^3 + \cdots$, which is geometric series.

The generalized hypergeometric function with p numerator and q denominator parameters is defined by, ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$

$$ = {}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_1, \dots, b_q \end{matrix}; z\right) = \sum_{r=0}^{\infty} \frac{(a_1)_r, (a_2)_r, \dots, (a_p)_r}{(b_1)_r, (b_2)_r, \dots, (b_q)_r} \frac{z^r}{r!} $$

3. *Product of hypergeometric series*

Generalization of Gauss functions obtained by considering the product of two Gauss functions, i.e.

$$_2F_1(a,b;c;x)\ {}_2F_1\ (a',b';c';y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}.$$

led to distinct possibilities of new functions. One such possibility, however gives us the double series,

$$\sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}.$$

which is simply the Gaussian series $\ {}_2F_1[a,b;c;x+y]$

4. *Gamma Function*

Definition: Let x be any positive number then the definite integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1} \exp(-t)\, dt, \ \ x > 0$$

Apart from the elementary transcendental functions such as the exponential and trigonometric functions and their inverses, the Gamma function is probably the most important transcendental function. It was defined by Euler to interpolate the factorials at non-integer arguments.

Following Euler, we define, $\Gamma(x) = \int_0^{\infty} t^{x-1} \exp(-t)\, dt, \ \ x > 0$

and call it the Gamma function.

This improper integral exists for complex z ∈ C with Re z > 0 (or, if you prefer only to think of real variables, for real z > 0). Using integration by parts, we get the fundamental functional equation

$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t}| + x\int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$

From the initial value

$\Gamma(1) = \int_0^\infty e^{-t} dt = e^{-t}| = 1.$

it follows further by induction that

- $\Gamma(x+1) = x!$, where x is a positive integer
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Beta Function

Definition: The Beta function denoted by $\beta(x,y)$ is defined as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \qquad x, y > 0$$

- $\beta(x,y) = \beta(y,x)$
- $\beta(x,y) = 2\int_0^{\frac{\pi}{2}} sin^{2x-1}\theta cos^{2y-1}\theta\, d\theta$
- $\int_0^{\frac{\pi}{2}} sin^x\theta cos^y\theta\, d\theta = \frac{1}{2}\beta\left(\frac{x+1}{2}, \frac{y+1}{2}\right)$

5. ***Relationship between Gamma and Beta functions***

$$\beta(x,y) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

6. ***Regular and Irregular singularities of a Differential Equations***

Singular points are classified as regular or irregular in the following way. If $P(z)$ or $Q(z)$ has a singularity at z_0 so that $u''(z_0)$ cannot be obtained for constructing the Taylor series of u(z), then the differential equation has a regular singularity if and only if both $(z - z_0)P(z)$ and $(z - z_0)^2 Q(z)$ are analytic at z_0. Otherwise, the singularity at $z = z_0$ is irregular.

7. ***Hypergeometric Equation in term of Differential Equation***

All of the differential equations we encounter in this book have at most three singularities. For these equations, the functions P(z) and Q(z) in Eq. (14) are rational. By an appropriate change of variable, these singularities may be transformed to three points $(0,1,\infty)$, in which case the differential equation takes the form. Hypergeometric equation: solution around z = 0, the hypergeometric series,

$f(a, b; c; z) = \sum_0^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$ satisfy the second order differential equation.

$$z(1-z)F'' + [c - (a+b+1)]F' - abF = 0 \qquad \text{(i)}$$

known as the hypergeometric equation.

This is Gauss's hypergeometric equation. The parameters a, b, and c are independent of z and, in general, may be complex.

In applying the series method, we assume a solution of the form, $u(z) = \sum_{n=0}^\infty a_n z^{n+s}$

and substitute this series into Eq. (i) to obtain

$$(z - z^2)\sum_{n=0}^{\infty} a_n(n+s)(n+s-1)z^{n+s-2}$$

$$+ [c - (a+b+1)z]\sum_{n=0}^{\infty} a_n(n+s)z^{n+s-1} - ab\sum_{n=0}^{\infty} a_n z^{n+s} = 0$$

Now let us collect terms with powers of z which look the same. Then

$$a_{n+1} = \frac{(n+a)(n+b)}{(n+c)(n+1)} a_n$$

The first few of these coefficients are,

$a_1 = \frac{ab}{c} a_0,$

$a_2 = \frac{(a+1)(b+1)}{2(c+1)} a_1 = \frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)} a_0,$

$a_3 = \frac{(a+2)(b+2)}{3(c+2)} a_2 = \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot c(c+1)(c+2)} a_0,$

If we continue the pattern, the nth coefficient is

$a_3 = \frac{(a+2)(b+2)}{3(c+2)} a_2 = \frac{a(a+1)\ \ldots\ (a+n-1)b(b+1)\ldots(b+n-1)}{n!c(c+1)\ldots(c+n-1)} a_0,$

So, as a solution to Eq. (i),

$u(z) = a_0\left[1 + \sum_{n=0}^{\infty} \frac{a(a+1)(a+2)\ \ldots\ (a+n-1)b(b+1)(b+2)\ldots(b+n-1)}{n!c(c+1)\ldots(c+n-1)} z^n\right],$

By using Pochhammer symbol, the expression in square brackets will be

$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n,$ (ii)

The function $F(a,b;c;z)$ defined by this series is the hypergeometric function.

8. ***Applications of Hypergeometric Functions***

The Simple Pendulum:

For an application of the hypergeometric function consider the exact solution of the simple pendulum problem. A simple pendulum consists of a point mass m attached to one end of a massless rod of length L. The other end of the rod is fixed at a point such that the system can swing freely under the influence of gravity.

The two forces exerted on the mass are the weight mg and the tension T in the rod. On resolving the equation of motion (Newton's second law) into two components

we have, centripetal: $T - mgcos\theta = \frac{mv^2}{L}$,

tangential: $-mgsin\theta = \frac{md^2}{dt^2}(L\theta)$

By rearranging the tangential equation, we get

$\frac{d^2\theta}{dt^2} + \frac{g}{L}sin\theta = 0$, second order differential equation. After further calculation,

The complete elliptic integral of the second kind is related to the hypergeometric function by

$$E(K) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 sin^2\emptyset}\, d\emptyset = \frac{\pi}{2} F\left(\frac{-1}{2}, \frac{1}{2}; 1; k^2\right)$$

The hypergeometric Function are used to solve,

a. One-Dimensional Harmonic Oscillator problem.

b. Cylinder Wave Guide Problem

c. The Vibrating Membrane Problem, etc

9. ***Some important deductions of the hypergeometric functions***

We know that,

$$2^{F_1}(\alpha,\beta;\gamma;z) = 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \cdots$$

- $F(*;*;z) = e^z$
- $F(-a;*;z) = (1-z)^a$
- $F\left(*,\frac{1}{2};-\frac{1}{4}z^2\right) = \cos(z)$
- $zF\left(*,\frac{3}{2};-\frac{1}{4}z^2\right) = \sin(z)$
- $zF(1,1;2;z) = \log(1-z) = -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right)$

10. ***Some Important Results about Generalized Hypergeometric and Confluent Hypergeometric Functions***

Recently, some generalizations of the generalized Gamma, Beta, Gauss hypergeometric and confluent hypergeometric functions have been introduced in literature. The nth derivative of $z^s F^{(\alpha,\beta)}(a,b;\ c;\ z)$ with respect to the variable z in a closed formula of hypergeometric function itself is obtained.

In mathematics, there are several special functions that are of particular significance and are used in many applications. Some of special functions find applications in such diverse areas as astrophysics, fluid dynamics and quantum physics. Examples of such well-known functions are the Gamma, Beta and hypergeometric functions. Next, extensions Gauss hypergeometric function (GHF) and confluent hypergeometric function (CHF) have been extensively studied inserting a regularization factor $e^{\frac{-p}{t}}$.

The following extension of the gamma function is introduced

$$\Gamma_p(x) = \int_0^\infty t^{x-1} \exp\left(-t - \frac{p}{t}\right) dt, \; Re(p) > 0. \tag{1}$$

The extension of Euler's beta function is considered in the following form

$$\beta_p(x) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(\frac{p}{t(1-t)}\right) dt, \; Re(p) > 0.$$

$$Re(p) > 0, Re(x) > 0, Re(y) > 0, \tag{2}$$

and

$$\Gamma_0(x) = \Gamma(x) \; and \; \beta_0(x,y) = \beta(x,y)$$

Using (2) used $\beta_p(x,y)$ to extent the hypergeometric function, known as the extended Gauss hypergeometric function (EGHF), as follows:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p(b+n,c-b)}{\beta(b,c-b)} \frac{z^n}{n!},$$

$$p \geq 0, Re(b) > 0, \tag{3}$$

where $(a)_n$ denotes the pochhammer symbol define by

$$(a)_n = \frac{\boldsymbol{\Gamma(a+n)}}{\boldsymbol{\Gamma(a)}}$$

$$= \begin{cases} 1, n = 0; \; a \in \mathbb{C}/\{0\} \\ a(a+1)(a+2) \ldots (a+n-1), \; n \in \mathbb{N}, a \in \mathbb{C} \end{cases}$$

the integral representation of Euler's type function is

$$F_p(a,b;c;z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left(\frac{-p}{t(1-t)}\right) dt,$$

$$p \geq 0 \; and \; |\arg(1-z)| < \pi < p; \; Re(c) > Re(b) > 0, \qquad (4)$$

the extended hypergeometric functions (ECHF) is defined as

$$\varphi_p(b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p(b+n,c-b)}{\beta(b,c-b)} \frac{z^n}{n!},$$

$$p \geq 0, Re(c) > Re(b) > 0, \qquad (5)$$

The following generalized Euler's gamma function (GEGF) is defined is

$$\Gamma_p{}^{(\alpha,\beta)}(x) = \int_0^1 t^{x-1} {}_1F_1\left(\alpha,\beta;-t-\frac{p}{t}\right) dt,$$

$$Re(\alpha) > 0, Re(\beta) > 0, Re(p) > 0, Re(x) > 0. \qquad (6)$$

While, the generalized Euler's beta function (GEBF) is given by

$$\beta_p{}^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right) dt,$$

$$Re(p) > 0, Re(x) > 0, Re(y) > 0, Re(\alpha) > 0, Re(\beta) > 0. \qquad (7)$$

The generalized (Gauss, resp. confluent) hypergeometric function (GGHF, resp., GCHF) are

defined by, $$F_p{}^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p{}^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)} \frac{z^n}{n!}, \qquad (8)$$

And $${}_1F_1{}^{(\alpha,\beta;p)}(b;c;z) = \sum_{n=0}^{\infty} \frac{\beta_p{}^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)} \frac{z^n}{n!}, \qquad (9)$$

and the corresponding integral representations are given by,

$$F_p(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}dt,$$

$$\text{Re}(p) \geq 0 \; and \; |\arg(1-z)| < \pi < p;\; Re(c) > Re(b) > 0, \quad (10)$$

and $${}_1F_1(b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right)dt,$$

$$\text{p} \geq 0, Re(c) > Re(b) > 0, \quad (11)$$

The generalized hypergeometric function with p numerator and q denominator parameters is defined by ${}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z)$

$$= {}_pF_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_1,\ldots,b_q\end{matrix};z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_r,(a_2)_r,\ldots,(a_p)_r}{(b_1)_r,(b_2)_r,\ldots,(b_q)_r}\frac{z^r}{r!},$$

Theorem 1. For the generalized Gauss hypergeometric function, Prove that:

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = (-1)^s \frac{a_n b_n}{c_n} \sum_{w=0}^{\infty} (a+n)_w \frac{1}{(1-a-w-n)_s} *$$

$$* \frac{1}{(w+n+1)_{(-s)}} \frac{\beta_p{}^{(\alpha,\beta)}(b+w+n-s,c-b)}{\beta(b+n,c-b)} \frac{z^w}{w!} \tag{12}$$

Proof. Substitute (8) into the left hand side of (12), we have

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = z^s \sum_{r=0}^{\infty} (a)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)} \frac{z^r}{r!}$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)} \frac{z^{r+s}}{r!}$$

$$= \sum_{r+s=n}^{\infty} (a)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)} (r+s)(r+s-1)\ldots(r+s-n+1) \frac{z^{r+s-n}}{r!}$$

$$= \sum_{r+s=n}^{\infty} (a)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)} \left\{\frac{(r+s)!}{(r+s-n)!}\right\} \frac{z^{r+s}}{r!}$$

Writing r + s – n = w, gives

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \sum_{w=0}^{\infty} (a)_{w+s-n} \frac{\beta_p{}^{(\alpha,\beta)}(b+w+s-n,c-b)}{\beta(b,c-b)} \left\{\frac{(w+n)!}{(w+s-n)!}\right\} \frac{z^w}{w!}$$

Using a deduction of pochammer symbol,

$$(a)_{w+n} = (a)_n (a+n)_w \ , \quad (a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k} \ ,$$

and $(a)_{w+n-s} = \frac{(-1)^s (a)_{n+w}}{(1-a-w-n)_s}$, we get

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = (-1)^s \sum_{w=0}^{\infty} \frac{(a)_n (a)_{n+w}}{(1-a-w-n)_s} \frac{{\beta_p}^{(\alpha,\beta)}(b+w+n-s,c-b)}{\beta(b,c-b)} \frac{\Gamma(w+n+1)}{\Gamma(w+n-s+1)} \frac{z^w}{w!}$$

$$= (-1)^s (a)_n \sum_{w=0}^{\infty} \frac{(a)_{n+w}}{(1-a-w-n)_s} \frac{{\beta_p}^{(\alpha,\beta)}(b+w+n-s,c-b)}{\beta(b,c-b)} \frac{\Gamma(w+n+1)}{\Gamma(w+n-s+1)} \frac{z^w}{w!}$$

By making use of formula $\beta(b,c-b) = \frac{b_n}{c_n}\beta(b+n,c-b)$ [1], we have

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = (-1)^s \frac{a_n b_n}{c_n} \sum_{w=0}^{\infty} (a+n)_w \frac{1}{(1-a-w-n)_s} *$$

$$\frac{1}{(w+n+1)_{(-s)}} \frac{{\beta_p}^{(\alpha,\beta)}(b+w+n-s,c-b)}{\beta(b+n,c-b)} \frac{z^w}{w!},$$

The particular expression for the derivatives of GGHF can be obtained as special cases from formula (7).

These are given in the following corollaries:

Corollary 1. Substitution of $s = 0$ and $n = 1$ into (12), then we get the 1st derivative of GGHF, $\frac{d}{dz}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{ab}{c} F_p^{(\alpha,\beta)}(a+1,b+1;c+1;z)$.

Corollary 2. Substitution of $s = 0$, into (12), make the nth derivative of GGHF as,

$$\frac{d}{dz}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{a_n b_n}{c_n} F_p^{(\alpha,\beta)}(a+n,b+n;c+n;z).$$

Theorem 2. For the generalized Gauss hypergeometric function (GGHF), the integral,

$$\int_0^1 x^{n-1}(1-x)^{m-1}F_p{}^{(\alpha,\beta)}(a,b;c;kx)dx = \ \beta(n,m)\sum_{r=0}^{\infty}\frac{(a)_r(n)_r}{(n+m)_r}\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}, \qquad (13)$$

Proof. Using (8) with relation (13), gives

$$\int_0^1 x^{n-1}(1-x)^{m-1}\sum_{n=0}^{\infty}(a)_n\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(kx)^r}{r!}\ dx$$

$$=\sum_{n=0}^{\infty}(a)_r\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}\int_0^1 x^{n-1}(1-x)^{m-1}\,dx,$$

$$=\sum_{n=0}^{\infty}(a)_r\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}\ \beta(n+r,m),$$

$$=\sum_{n=0}^{\infty}(a)_r\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}\left[\frac{\Gamma(n+r)\Gamma(m)}{\Gamma(m+n+r)}\right],$$

$$=\beta(n,m)\sum_{n=0}^{\infty}(a)_r\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}\left[\frac{\Gamma(n+r)}{\Gamma(m+n+r)}\frac{\Gamma(m+n)}{\Gamma(n)}\right],$$

$$=\beta(n,m)\sum_{n=0}^{\infty}\frac{(a)_r(n)_r}{(n+m)_r}\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!},$$

This complete the proof of the theorem.

Corollary 1. The integral of the classical (GHF), which is obtained by taking p = 0, is

$$\int_0^1 x^{n-1}(1-x)^{m-1}2^F{}_1(a,b;c;kx)dx = \ \beta(n,m)3^F{}_2\left({a,b,n \atop n+m};k\right).$$

Corollary 2. The integral of (GCHF) is given by,

$$\int_0^1 x^{n-1}(1-x)^{m-1}F_p{}^{(\alpha,\beta)}(b;c;kx)dx = \beta(n+r,m)\sum_{n=0}^{\infty}\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!},$$

Theorem 3. For the generalized Gauss hypergeometric function (GGHF), the integral,

$$\int_0^1 x^{\mu-1}e^{-m^2x^2}F_p^{(\alpha,\beta)}(a,b;c;\pm n^2x^2)dx = \frac{\Gamma(\frac{\mu}{2})}{2m^{\mu}}\sum_{r=0}^{\infty}(a)_r\left(\frac{\mu}{2}\right)_r\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{\left(\pm\frac{n^2}{x^2}\right)}{r!}, \quad (14)$$

Proof. By using (10), we have, $\int_0^1 x^{\mu-1}e^{-m^2x^2}F_p^{(\alpha,\beta)}(a,b;c;\pm n^2x^2)dx$

$$= \frac{1}{\beta(b,c-b)}\int_0^{\infty}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-(\pm n^2x^2)t)^{-a}x^{\mu-1}e^{-m^2x^2}{}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right)dxdt$$

However, $(1-(\pm n^2x^2)t)^{-a} = \sum_{r=0}^{\infty}\frac{(a)_r}{r!}(\pm n^2x^2)t^r$, so

$$= \frac{1}{\beta(b,c-b)}\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\int_0^{\infty}\int_0^1 t^{b-1}(1$$

$$-t)^{c-b-1}(\pm 1)^r n^{2r}e^{-m^2x^2}x^{\mu+2r-1}{}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right)dxdt$$

Using equation (7), $\sum_{r=0}^{\infty}\frac{(a)_r}{\beta(b,c-b)}\int_0^{\infty}e^{-m^2x^2}x^{\mu+2r-1}dx\beta_p^{(\alpha,\beta)}(b+r,c-b)\frac{(\pm n^2)^r}{r!}$.

Using the definition of Gamma function, $\Gamma(x) = \int_0^{\infty}t^{x-1}e^{-t}dt$, and letting $m^2x^2 = w$,

$$\int_0^{\infty}e^{-m^2x^2}x^{\mu+2r-1}dx = \frac{1}{2m}\int_0^{\infty}e^{-w}\left(\frac{\sqrt{w}}{m}\right)^{\mu+2r-1}\frac{1}{\sqrt{w}}dx = \frac{\frac{1}{2}}{m^{u+2r}}\Gamma\left(\frac{\mu}{2}+r\right),$$

then, $\int_0^1 x^{\mu-1}e^{-m^2x^2}F_p^{(\alpha,\beta)}(a,b;c;\pm n^2x^2)dx$

$$= \frac{1}{2}\sum_{r=0}^{\infty}\frac{(a)_r}{\beta(b,c-b)}\frac{1}{m^{u+2r}}\Gamma\left(\frac{\mu}{2}+r\right)\beta_p^{(\alpha,\beta)}(b+r,c-b)\frac{(\pm n^2)^r}{r!},$$

$$= \frac{1}{2}\sum_{r=0}^{\infty}\frac{(a)_r}{\beta(b,c-b)}\frac{1}{m^{u+2r}}\left(\frac{\mu}{2}\right)_r\Gamma\left(\frac{\mu}{2}\right)\beta_p^{(\alpha,\beta)}(b+r,c-b)\frac{(\pm n^2)^r}{r!},$$

$$= \frac{1}{2}\sum_{r=0}^{\infty} \frac{(a)_r}{m^u} (\frac{\mu}{2})_r \Gamma\left(\frac{\mu}{2}\right) \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)} \frac{(\frac{\pm n^2}{m^2})^r}{r!},$$

$$= \frac{\Gamma\left(\frac{\mu}{2}\right)}{2m^u}\sum_{r=0}^{\infty} (a)_r (\frac{\mu}{2})_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)} \frac{(\frac{\pm n^2}{m^2})^r}{r!},$$

Corollary 1. $\int_0^1 x^{\mu-1} e^{-m^2x^2} F_p{}^{(\alpha,\beta)}(a, b; c; \pm n^2x^2)dx$

$$= \frac{\Gamma\left(\frac{\mu}{2}\right)}{2m^u}\sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r} (\frac{\mu}{2})_r \frac{(\frac{\pm n^2}{m^2})^r}{r!},$$

$$= \frac{\Gamma\left(\frac{\mu}{2}\right)}{2m^u} 3^F{}_2 \left({a, b, \frac{\mu}{2} \atop c} ; \frac{\pm n^2}{m^2} \right)$$

Corollary 2. $\int_0^1 x^{\mu-1} e^{-m^2x^2} F_p{}^{(\alpha,\beta)}(a, b; c; \pm n^2x^2)dx$

$$= \frac{1}{2m^u} \sum_{r=0}^{\infty} \Gamma\left(\frac{\mu}{2} + r\right) \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)} \frac{(\frac{\pm n^2}{m^2})^r}{r!}$$

Corollary 3. $\int_0^1 x^{\mu-1} e^{-m^2x^2} p^F{}_q \left({(a)_1(a)_2 \ldots\ldots (a)_p \atop (b)_1(b)_2 \ldots\ldots (b)_q} ; \pm n^2x^2 \right) dx$

$$= \frac{\Gamma\left(\frac{\mu}{2}\right)}{2m^u} p + 1^F{}_q \left({(a)_1(a)_2 \ldots\ldots (a)_p, \frac{\mu}{2} \atop (b)_1(b)_2 \ldots\ldots (b)_q} ; \pm \frac{n^2}{m^2} \right) dx,$$

Theorem 4. For the generalized Gauss hypergeometric function (GGHF),

$$\int_0^1 x^{n-1}(1-x)^{m-1} F_p{}^{(\alpha,\beta)}\left(a, b; c; \frac{1-x}{2}\right) dx = \sum_{r=0}^{\infty} (a)_r\, \beta(n, m+r) \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)} \frac{(\frac{1}{2})^r}{r!}$$

Proof. $$\left(1-\frac{(1-x)}{2}t\right)^{-a}=\sum_0^\infty \frac{a_r}{r!}\frac{(1-x)^r}{2}\,t^r,$$

using (10),

$$F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\;\sum_{r=0}^\infty\frac{(a)_r}{r!}\left(\frac{1-x}{2}\right)^r t^r\,{}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right)dt,$$

$$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^\infty\frac{(a)_r}{2^r r!}\int_0^1\int_0^1 x^{n-1}(1-x)^{m-1}t^{b+r-1}(1-t)^{c-b-1}\;{}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right)dxdt,$$

$$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^\infty\frac{(a)_r}{2^r r!}\int_0^1 x^{n-1}(1-x)^{m+r-1}dx\int_0^1 t^{b+r-1}(1-t)^{c-b-1}\;{}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right)dt,$$

Using (7), $\beta_p^{(\alpha,\beta)}(b+r,c-b)=\int_0^1 t^{b+r-1}(1-t)^{c-b-1}{}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right)dt,$

which gives, $$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^\infty(a)_r\,\beta(n,m+r)\,\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(\frac{1}{2})^r}{r!}.$$

Corollary 1. $$\int_0^1 x^{n-1}(1-x)^{m-1}F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\beta(n,m)\sum_{r=0}^\infty\frac{(a)_r(b)_r}{(c)_r}\frac{(m)_r}{(m+n)_r}\frac{\left(\frac{1}{2}\right)^r}{r!},$$

$$=\beta(n,m)\,{}_3F_2\left(\begin{matrix}a,b,m\\ m+n,c\end{matrix};\frac{1}{2}\right).$$

Corollary 2. $$\int_0^1 x^{n-1}(1-x)^{m-1}\varphi_p^{(\alpha,\beta)}\left(b;c;\frac{1-x}{2}\right)dx$$

$$=\sum_{r=0}^\infty(a)_r\,\beta(n,m+r)\,\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(\frac{1}{2})^r}{r!}.$$

Corollary 3. $$\int_0^1 x^{n-1}(1-x)^{m-1}{}_pF_q\left(\begin{matrix}(a)_1(a)_2\ldots\ldots(a)_p\\ (b)_1(b)_2\ldots\ldots(b)_q\end{matrix};\frac{1-x}{2}\right)dx$$

$$=\beta(n,m)\,{}_{p+1}F_{q+1}\left(\begin{matrix}(a)_1(a)_2\ldots\ldots(a)_p,m\\ (b)_1(b)_2\ldots\ldots(b)_q,n+m\end{matrix};\frac{1}{2}\right)dx$$

Theorem 5. For the generalized Gauss hypergeometric function (GGHF),

$$\int_0^1 (1-x^2)^{m-1} F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right) dx \; = \sum_{r=0}^{\infty} 2^{2m+r-2}\, \beta(m+r,m) \frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)} \frac{(a)_r (\frac{1}{2})^r}{r!}$$

Proof. Using (10),

$$F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\; {}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right) \sum_{r=0}^{\infty} \frac{(a)_r}{r!} \left(\frac{1-x}{2}\right)^r t^r dt,$$

Then, $\int_0^1 (1-x^2)^{m-1} F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right) dx \; = \frac{1}{\beta(b,c-b)} \int_0^1 \int_0^1 (1-x^2)^{m-1} t^{b-1} (1-$

$t)^{c-b-1}\; {}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right) \sum_{r=0}^{\infty} \frac{(a)_r}{r!} \left(\frac{1-x}{2}\right)^r t^r dtdx,$

Let $\mathrm{x} = \cos\theta$, $1-\mathrm{x} = 2\sin^2(\frac{\theta}{2})$, and $1+x = 2\cos^2(\frac{\theta}{2})$. Then,

$\int_0^1 (1-x^2)^{m-1} F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right) dx \; =$

$\frac{1}{\beta(b,c-b)} \sum_{r=0}^{\infty} \frac{(a)_r}{2^r r!} \int_0^{\frac{\pi}{2}} \int_0^1 (2\sin^2(\frac{\theta}{2}))^r \left(2\sin^2(\frac{\theta}{2}) 2\cos^2(\frac{\theta}{2})\right)^{m-1} \sin\theta\; t^{b+r-1}(1-$

$t)^{c-b-1}\; {}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right) d\theta dt,$

$= \frac{1}{\beta(b,c-b)} \sum_{r=0}^{\infty} \int_0^{\frac{\pi}{2}} \int_0^1 2^{2m+r-1} \sin^{2m+2r-1}\left(\frac{\theta}{2}\right) \cos^{2m-1}\left(\frac{\theta}{2}\right) t^{b+r-1}(1-$

$t)^{c-b-1}\; {}_1F_1\left(\alpha,\beta;\frac{-p}{t(1-t)}\right) \frac{(a)_r}{2^r r!} d(\frac{\theta}{2}) dt,$

$\int_0^1 (1-x^2)^{m-1} F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right) dx$

Using (2) and (7),

$$= \sum_{r=0}^{\infty} 2^{2m+r-2}\, \beta(m+r,m) \frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)} \frac{(a)_r (\frac{1}{2})^r}{r!},$$

11. Conclusion and future scope

In this project, the nth derivative of Gauss hypergeometric functions and confluent hypergeometric functions in term of hypergeometric functions themselves is expressed. This project report has dealt with formulae expressing explicitly the generalized hypergeometric functions in term of classical hypergeometric function itself. Some results are obtained and hoping to extend the results for special function in the near future.

12. References

[1] S. I. El-Soubhy. Notes on Generalized Hypergeometric and Confluent Hypergeometric Functions, International Journal of Mathematical Analysis and Applications. Vol. 2, No. 3, 2015, pp. 47-51.

[2] M.A. Chaudhry, A. Qadir, H.M. Srivastava, R.B Paris, Extended Hypergeometric and Confluent Hypergeometric Functions, Appl. Math. Comput. 159,(2004), pp. 589-602.

[3] S. I. El-Soubhy. Notes on Generalized Hypergeometric and Confluent Hypergeometric Functions. International Journal of Mathematical Analysis and Applications. Vol. 2, No. 3, 2015, pp 47-51.

[4] M.D. Raisinghania, Hypergeometric Function, Ordinary and Partial Differential Equations, S. Chand and Company Limited, 19^{th} Edition, 2017, pp 14.1 - 14.18

[5] S. C. Malik, Savita Arora, Beta and Gamma Functions. Mathematical Analysis, New Age International (P) Publishers, 5^{th} Edition, 2017, pp. 839-845.

[6] James B. Seaborn F. John, J.E. Marsden, L. Sirovich, M. Golubitsky, w. Jager, Hypergeometric Functions and Their Applications, Springer Science + Business Media, LLC, pp 30-66.